Deformation of Schemes

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1 The Affine Case

Lemma 1.1. *Let $f : C \twoheadrightarrow R$ be an object of $A-Alg/R$ and let suppose I, the kernel of f, is a square-zero ideal in C. Then any ring homomorphism that is a section $s : R \to C$*[1] *will give a ring isomorphism $\sigma : R[I] \xrightarrow{\sim} C$ where $\sigma(r,i) = s(r) + i$ with inverse $\sigma^{-1}(c) = (f(c), c - sf(c))$*

Proposition 1.2. *Suppose we have a surjection $C \twoheadrightarrow R$ with square zero kernel I and suppose we have a ring automorphism of the form*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}"] & C \arrow[r] \arrow[d, "\varphi"] & R \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & I \arrow[r] & C \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

Then φ must be of the form $\varphi(i + r) = i + r + \delta(r)$ where $i \in I, r \in R$ and $\delta : R \to I$. Furthermore δ is a derivation.

Proof. We know $\varphi(i) = i$ while $\varphi(r) = r \mod I$ and so $\varphi(r) = r + \delta(r)$ for $\delta : R \to I$.

Proposition 1.3. *Suppose that $X = \operatorname{Spec} B$ is a smooth affine $R-$scheme and $\pi : R[I] \twoheadrightarrow R$ with kernel I. Then $\mathrm{Def}_X^{\mathrm{smooth}}(R, \pi)$ consists of one element.*

Proof. We first give an element of $\mathrm{Def}_X^{\mathrm{smooth}}(R[I], \pi)$. Consider $g : R[I] \to R[I] \otimes_R B$ where $(r,i) \mapsto (r,i) \otimes 1$ and where $R \to R[I]$ by $r \mapsto (r, 0)$, so is a section of π. As a result, we will have that $R \otimes_{R[I]} (R[I] \otimes_R B) = B$ as r goes to r going from right to left. Moreover note that g is flat as it arises from base change of a flat morphism. Thus $R[I] \to R[I] \otimes_R B \in \mathrm{Def}_X^{\mathrm{smooth}}(R, \pi)$. Also note that we have $R[I] \otimes_R B \cong B[I \otimes_R B]$ using the (injective) section $s_B : B \to R[I] \otimes_R B,\ b \mapsto 1 \otimes b$ and applying Lemma 1.1.

Now suppose that $D \in \mathrm{Def}_X^{\mathrm{smooth}}(R[I], \pi)$. so that the following diagram is co-Cartesian.

$$\begin{tikzcd} B & D \arrow[l] \\ R \arrow[u] & R[I] \arrow[l, "\pi"'] \arrow[u, "a"'] \end{tikzcd}$$

This means we have a $R-$linear ring isomorphism $\Phi : R \otimes_{R[I]} D \xrightarrow{\sim} B$. Writing $R = R[I]/I$, we obtain a surjective map $\Phi' : D \twoheadrightarrow D/ID \cong B$ whose kernel ID is square-zero. Because we have a section s_R of π, we can replace π with s_R and the diagram above still commutes. As a result we have the following solid commutative diagram

[1] Such a map may not exist, for example, consider $f : \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ where $f(1) = 1$. The kernel will be $p(\mathbb{Z}/p^2\mathbb{Z})$ which is a square-zero ideal. Any ring homomorphism back must send 1 to a multiple of p but then this is clearly not a section.

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] & B = D/I \\ R \arrow[u] \arrow[r, "a\circ s_R"'] & D \arrow[u, "\Phi'"'] \arrow[from=ul, dotted] \end{tikzcd}$$

But $R \to B$ is smooth and thus formally smooth and so we obtain a map $s : B \to D$ which commutes above, aka s is a section of Φ'. Because B is a flat R module, and D is a flat $R[I]$ module, we have that $I \otimes_R B = IB$ and $I \otimes_{R[I]} D = ID$ and so we have the following commutative diagram where the middle map is a ring homomorphism (similar to Lemma 1.1)

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_R B \arrow[r] \arrow[d, "id\otimes s(b)"] & B[I \otimes_R B] \arrow[r, "mod I\otimes_R B"] \arrow[d, "\Psi"] & B \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & I \otimes_{R[I]} D \arrow[r, "i\cdot d"] & D \arrow[r, "\Phi'"] & B \arrow[r] & 0 \end{tikzcd}$$

where $\Psi(b, i \otimes b') = s(b) + i \cdot s(b')$. We claim that $id \otimes s(b)$ is an isomorphism. Indeed one can check that $id \otimes \Phi'$ will be the reverse map and we have that $i \otimes d \mapsto i \otimes \Phi'(d) \mapsto i \otimes s(\Phi'(d)) = i \otimes d$ where the last step is because $s(\Phi'(d)) = d + \sum i' \cdot d'$ and I is square-zero. Hence we can apply 5 lemma to conclude. Like in Lemma 1.1 we can also show that the inverse map Ψ^{-1} is given by

$$\Psi^{-1}(d) = (\Phi'(d), (\mathrm{id} \otimes \Phi')(d - s \circ \Phi'(d)))$$

where we first write $d - s \circ \Phi'(d) \in \ker \Phi' \cong I \otimes_{R[I]} D$ and then apply $\mathrm{id} \otimes \Phi'$.

Lemma 1.4. *Let $q : B' \to B$ be a surjective homomorphism of $k-$algebras with square-zero kernel I*

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

Then we have that

(a) If $f, g : B \to B'$ are two sections of q, then $\theta = g - f$ is a $k-$derivation of R to I,

(b) Conversely, if $f : R \to B'$ is one section, and $\theta : R \to I$ is a derivation, then $g = f + \theta$ is another section of q.

Remark. Note that (b) says that $\mathrm{Der}_k(R, I)$ (a) acts on sections of q and since the operation is addition of functions, the action has to be free, while (a) says that the action of is transitive and so sections of q is a torsor for the action of $\mathrm{Der}_k(R, I)$.

2 General Case

Theorem 1
Assume X is a smooth R scheme and I is a flat $R-$module. Then there is a bijection

$$\mathrm{Def}_X^{\mathrm{smooth}}(R[I]) \xrightarrow{\sim} H^1(X, \mathcal{T}_{X/R} \otimes_R I)$$

Proof. Suppose we have a deformation $X' \in \mathrm{Def}_X(R[I])$. Then the following diagram is Cartesian

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] & \mathrm{Spec}\, R[I] \end{tikzcd}$$

Because the underlying topological space of X' is the same as X, given an open affine cover $\{U_k = \operatorname{Spec} B_k\}$ of X, we obtain an open affine cover $\{U'_k = \operatorname{Spec} D_k\}$ for X', each U'_k will fit into the top right corner of the above diagram. By Proposition 1.3 we can trivialize the deformation D_k, aka we have $R[I]$ linear ring isomorphisms

$$\varphi_k : R[I] \otimes_R B_k \to D_k$$

such that modulo I, φ_k will be the identity on B_k[2]. Now WLOG, assume that $U'_{kj} = U'_k \cap U'_j$ is a distinguished open for both U_k and U_j(See [Vakil] Proposition 5.3.1) and let $U_k = \operatorname{Spec} B_{kj}$, $U'_{kj} = \operatorname{Spec} D_{kj}$. It follows that both $\varphi_k|_{U'_{kj}}, \varphi_j|_{U'_{kj}} : R[I] \otimes_R B_{kj} \to D_{kj}$ induce the identity on B_{kj} being a localization and so $\varphi_j^{-1} \circ \varphi_k : R[I] \otimes_R B_{kj} \to R[I] \otimes_R B_{kj}$ is a ring automorphism satisfying the conditions in Proposition 1.2 and thus

$$\varphi_j^{-1} \circ \varphi_k(b, i \otimes b') = (b, \alpha_{kj}(b) + i \otimes b') \tag{1}$$

where $\alpha_{kj} \in \operatorname{Der}_R(B_{kj}, I \otimes_R B_{kj})$. But by definition,

$$\mathcal{T}_X \otimes_R I(B_{kj}) = \operatorname{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/R}, B_{kj}) \otimes_R I \overset{*}{=\!=} \operatorname{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/R}, I \otimes_R B_{kj}) = \operatorname{Der}_R(B_{kj}, I \otimes_R B_{kj})$$

where the equality $*$ is because for any $R-$module K, $K \otimes_R I = K \otimes_B (B \otimes_R I)$ and since I is a flat $R-$module, it follows that $B \otimes_R I$ is a flat B module as $R \to B$ is flat, and now let $N = B_{kj} \otimes_R I$ in

$$\operatorname{Ext}^p_B(M, B) \otimes_B N \cong \operatorname{Ext}^p_B(M, N)$$

(see Poincare duality in Hochschild cohomology). Thus $\alpha_{kj} \in H^0(B_{kj}, \mathcal{T}_X \otimes_R I)$. Since $\varphi_\ell^{-1} \circ \varphi_j \circ \varphi_j^{-1} \circ \varphi_k = \varphi_\ell^{-1} \circ \varphi_k$, it follows that

$$(b, \alpha_{j\ell}(b) + \alpha_{kj}(b) + i \otimes b') = (b, \alpha_{k\ell}(b) + i \otimes b')$$

so that the collection $\{\alpha_{kj}\}$ (which depend on the φ_k) is in $Z^1(\mathscr{U}, \mathcal{T}_X \otimes_R I)$. Two deformations are isomorphic if we choose different isomorphisms φ_k on each affine open. Each φ_k is defined using a section $s_k : B_k \to D_k$ of $\pi_k : D_k \to B_k$. Let φ'_k be defined using another section s'_k and φ'_j be defined using another section s'_j. Let $\theta_k = s'_k - s_k \in \operatorname{Der}_R(B_k, I \otimes_R B_k)$ and $\theta_j \in \operatorname{Der}_R(B_j, I \otimes_R B_j)$. One can then compute that $(\varphi'_k - \varphi_k)(b, i \otimes b') = \theta_k(b)$ while $(\varphi'_j)^{-1} - \varphi_j^{-1})(d) = (0, -\theta_j(\pi_j(d)))$.

$$\left((\varphi'_j)^{-1} \circ \varphi'_k - \varphi_j^{-1} \circ \varphi_k\right)(b, i \otimes b') = (0, \theta_k(b)|_{B_{kj}} - \theta_j(b)|_{B_{kj}}) = (0, \alpha'_{kj}(b) - \alpha_{kj}(b))$$

where we used that $\pi_j|_{B_{kj}} = \pi_k|_{B_{kj}}$. The term in the middle is exactly d of an element in $B^0(\mathscr{U}, \mathcal{T}_X \otimes_R I)$ and so we obtain

$$\begin{aligned} KS : \operatorname{Def}^{\text{smooth}}_X(R[I]) &\to H^1(X, \mathcal{T}_{X/R} \otimes_R I) \\ X' &\mapsto \{\alpha_{kj}\} \end{aligned}$$

Conversely, given a collection $\{\alpha_{kj}\}$, because they satisfy the cocycle condition, $\varphi_{jk} := \varphi_j^{-1} \circ \varphi_k$ as defined in Eq. (1) will satisfy another cocycle condition necessary to glue together trivial deformations on the affine open sets to form a scheme X'. One can check these are inverses to each other.

Remark. KS is called the Kodaira-Spencer map and $\{\alpha_{kj}\}$ is called the Kodaira-Spencer class of $X' \in \operatorname{Def}^{\text{smooth}}_X(R[I])$.

[2]The isomorphism was induced by a section.

3 Obstructions

Definition 3.1. *Given a functor $F : \mathrm{Art}_k \to \mathrm{Set}$ an obstruction theory for F taking values in a $k-$vector space V consists of the data, for each thickening $\varphi : A' \to A$ with kernel I, a map $\mathrm{ob}(\varphi) : F(A) \to V \otimes_k I$ such that*

(i) For $\eta \in F(A)$, $\mathrm{ob}(\varphi)(\eta) = 0 \iff \exists \eta' \in F(A')$ such that $F(\varphi)(\eta') = \eta$.

(ii) (naturality) Given a factoring of φ into a sequence of thickenings, $\begin{tikzcd} A' \arrow[r] \arrow[rr, bend left, "\varphi"] & B \arrow[r, "\psi"'] & A \end{tikzcd}$ *with the kernel of ψ equal to I/J we have that the following diagram commutes*

$$\begin{tikzcd} F(A) \arrow[r, "\mathrm{ob}(\varphi)"] \arrow[dr, "\mathrm{ob}(\psi)"'] & V \otimes_k I \arrow[d] \\ & V \otimes_k I/J \end{tikzcd}$$

Given $f : X' \to \operatorname{Spec} A$ a smooth morphism and $A' \twoheadrightarrow A$ a surjection of rings whose kernel J is square-zero, when can we find a lift X'' s.t. the following diagram is Cartesian

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X'' \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} A' \end{tikzcd}$$

Fix a affine cover $\{V_k = \operatorname{Spec} D_k\}$ of X' by affines. Applying Proposition 1.3 we can obtain a lifting V_k' over A'. Now suppose for each pair (j, k) we choose isomorphisms

$$\psi_{jk} : V_j'|_{V_{jk}} \to V_k'|_{V_{jk}}$$

lifting the map $V_j|_{V_{jk}} \to V_k|_{V_{jk}}$. Then it's not always true that the collection $\{\psi_{jk}\}$ will satisfy the cocycle condition. So on triple overlaps, define

$$C_{jk\ell} = \psi_{k\ell} \circ \psi_{jk} \circ \psi_{j\ell}^{-1}$$

This will be a A' linear automorphism of $V_\ell'|_{V_{jk\ell}} = \operatorname{Spec} A' \otimes_A D_{jk\ell}$ (again can assume that $V_{jk\ell} = \operatorname{Spec} D_{jk\ell}$ is affine) which reduces to the identity modulo J as X' is a scheme. Therefore we can apply Proposition 1.2 to

$$0 \to J \otimes_A D_{jk\ell} \to A' \otimes_A D_{jk\ell} \to D_{jk\ell} \to 0$$

to see that

$$C_{jk\ell}(d, j \otimes d') = (d, \beta_{jk\ell}(d) + j \otimes d')$$

where $\beta_{jk\ell} \in \mathrm{Der}_A(D_{jk\ell}, J \otimes_A D_{jk\ell}) = (\mathcal{T}_{X'/A} \otimes_A J)(D_{jk\ell})$.

Lemma 3.2. *(a) The collection $\{\beta_{jk\ell}\} \in Z^2(\mathscr{U}, \mathcal{T}_{X'/A} \otimes_A J)$.*

(b) If $\{\psi_{jk}'\}$ is any other choice of isomorphisms with corresponding $\{\beta_{jk\ell}'\}$ then $\{\beta_{jk}\} - \{\beta_{jk}'\} \in B^2(\mathscr{U}, \mathcal{T}_{X'/A} \otimes_A J)$.

Thus for any smooth $f : X' \to \operatorname{Spec} A$ we obtain a well defined class $o(f) \in H^2(X', \mathcal{T}_{X'/A} \otimes_A J)$.

Proposition 3.3. *There exists a lifting* $X'' \to \operatorname{Spec} A' \iff o(f) = 0$.

Proof. Being in $B^2(\mathscr{U}, \mathcal{T}_{X'/A} \otimes_A J)$ is essentially saying you are 1 up to automorphisms.

Example. Let $X' \in \operatorname{Def}_X^{\text{smooth}}(k[\epsilon])$, and consider the surjection $k[x]/(x^3) \twoheadrightarrow k[\epsilon] = k[x]/(x^2)$ with kernel $J = (x^2)/(x^3) = k$. Then we have that the obstruction to lifting X' to a higher order deformation lives in

$$H^2(X', T_{X'/k[\epsilon]} \otimes_{k[\epsilon]} {}^{(x^2)}/_{(x^3)}) \cong H^2(X, T_{X/k} \otimes_k k)$$

where the isomorphism above is because on affine pieces for any $k[\epsilon]$ module M

$$M \otimes_{k[\epsilon]} (x^2)/(x^3) \cong M/(x) \otimes_{k[\epsilon]/(x)} (x^2)/(x^3)$$

4 Examples

Theorem 2 (Computation of Cohomology for Curves)
Let C be a smooth projective curve, $T = T_X$ the tangent sheaf and $K = \Omega^1_C$ the canonical sheaf. Then we have

	deg	h^0	h^1	h^2
K	$2g-2$	g	*1*	*0*
T	$2-2g$	ϵ	$\epsilon + 3g - 3$	*0*

where $\epsilon = 0$ if $g \geq 2$, $\epsilon = 1$ if $g = 1$, and $\epsilon = 3$ if $g = 0$.

Proof. Plugging in $D = K$ in Riemann Roch we obtain

$$h^0(K) - 1 = \deg(K) - g + 1$$

and note that $h^0(K) = g$ as this is the dimension of the differential forms on C which is the rank of the first homology group and thus is g and so $\deg(K) = 2g-2$. As T is the dual of K we have that $\deg(T) = 2-2g$. By Riemann-Roch we will have that

$$h^1(T) = \epsilon - \deg(T) + g - 1 = 3g - 3 + \epsilon$$

so it remains to compute $h^0(T)$. Notice that for $g \geq 2$ we have that $\deg T < 0$ and therefore $h^0(T) = 0$. For $g = 1$, notice that $h^0(K) = 1$ and $\deg(K) = 0$. It follows that $K = \mathcal{O}_C$, indeed $h^0(K) = 1$ means we have a non-zero (holomorphic) section s of K and such a section cannot have poles. $\deg(K) = 0$ will then imply it can't have any zeros as well and so gives our desired isomorphism. It follows that $T = \mathcal{O}_X$ and so $\epsilon = 1$. For $g = 0$, note that $\deg(K) < 0$ and so $\deg(K^2) < 0$ and thus by Serre duality $h^1(T) = h^0(K^2) = 0$ and so $\epsilon = 3$ as desired.

Theorem 4.1. $h^1(\mathbb{P}^n_A) = 0$.

Proof. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$. Consider the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

From the LES the following is exact

$$H^1(\mathcal{O}(1)^{\oplus n+1}) \to H^1(\mathcal{T}_{\mathbb{P}^n}) \to H^2(\mathcal{O})$$

and the left and rightmost terms are 0 since non-negative degree line bundles have no higher cohomology.